

# MAPPING PROPERTIES OF WEIGHTED BERGMAN PROJECTION OPERATORS ON REINHARDT DOMAINS

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**ABSTRACT.** We show that on smooth complete Reinhardt domains, weighted Bergman projection operators corresponding to exponentially decaying weights are unbounded on  $L^p$  spaces for all  $p \neq 2$ . On the other hand, we also show that the exponentially weighted projection operators are bounded on Sobolev spaces on the unit ball.

## 1. INTRODUCTION

Let  $\Omega$  be a bounded domain in  $\mathbb{C}^n$  and let  $\lambda$  be a positive continuous function on  $\Omega$ . We denote the standard Lebesgue measure by  $dV(z)$  and we consider  $\lambda$  as a weight function on  $\Omega$ . The space of square integrable functions on  $\Omega$  with respect to the measure  $\lambda(z)dV(z)$  is denoted by  $L^2(\Omega, \lambda)$ . The weighted inner product and the corresponding norm are defined in the usual way. The space of square integrable holomorphic functions on  $\Omega$  is denoted by  $L_a^2(\Omega, \lambda)$ . Since the weight is continuous and positive,  $L_a^2(\Omega, \lambda)$  is a closed subspace of  $L^2(\Omega, \lambda)$  (see [PW90]). The weighted Bergman projection operator  $\mathbf{B}_\Omega^\lambda$  is the orthogonal projection operator from  $L^2(\Omega, \lambda)$  onto  $L_a^2(\Omega, \lambda)$ . By the Riesz representation theorem,  $\mathbf{B}_\Omega^\lambda$  is an integral operator of the form

$$\mathbf{B}_\Omega^\lambda f(z) = \int_\Omega B_\Omega^\lambda(z, w) f(w) \lambda(w) dV(w)$$

where the kernel  $B_\Omega^\lambda(z, w)$  is called the weighted Bergman kernel. As an integral operator, mapping properties of  $\mathbf{B}_\Omega^\lambda$  on  $L^p(\Omega, \lambda)$  for  $p \neq 2$  are also of interest. In particular, for a given pair  $\Omega$  and  $\lambda$ , one can ask the question for which  $p \in (1, \infty)$  is the operator  $\mathbf{B}_\Omega^\lambda$  bounded on  $L^p(\Omega, \lambda)$ .

This question has been addressed in many settings and we refer the reader to [LS04, Lig89, CD06, KP07, CL97, BŞ12] and the references therein. One of the well studied settings is the case when  $\Omega$  is the unit disk  $\mathbb{D}$  in  $\mathbb{C}^1$ . Two extreme cases stand out on  $\mathbb{D}$ . Let  $\rho(z) = |z|^2 - 1$  be the standard defining function\* for  $\mathbb{D}$ .

- (Polynomial decay [FR75]) If  $\lambda = (-\rho)^k$  for some  $k > 0$  then the weighted Bergman projection operator  $\mathbf{B}_\mathbb{D}^\lambda$  is bounded on  $L^p(\mathbb{D}, \lambda)$  for all  $p \in (1, \infty)$ .
- (Exponential decay [Dos04, Zey13a]) On the other hand, if  $\lambda = \exp\left(\frac{1}{\rho}\right)$  then the weighted Bergman projection operator  $\mathbf{B}_\mathbb{D}^\lambda$  is bounded on  $L^p(\mathbb{D}, \lambda)$  if and only if  $p = 2$ .

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\*Both of the results hold for more general defining functions.

The change in  $\lambda$ , from polynomial decay to exponential decay, causes a drastic change in  $L^p$  boundedness of weighted Bergman projections.

We are interested in whether the same phenomenon happens on domains in higher dimensions when we change the weight from polynomial to exponential decay.

**1.1.  $L^p$  irregularity.** We start our investigation on smooth complete Reinhardt domains. In this setting, the polynomial decay case (under some additional conditions on  $\Omega$ ) follows from a more general theorem in [CDM13, CDM14]. Namely, if  $\Omega$  is a smooth bounded Reinhardt pseudoconvex domain of finite type in  $\mathbb{C}^2$ ,  $\rho$  is a defining function for  $\Omega$  and  $\lambda = (-\rho)^q$  where  $q$  is a non-negative rational number then the weighted Bergman projection operator  $\mathbf{B}_\Omega^\lambda$  is bounded on  $L^p(\Omega, \lambda)$  for all  $p \in (1, \infty)$ . See [CDM13, CDM14] and also [BG95] for the case  $q$  is a natural number and [MS94] for the case  $q = 0$ .

This result is analogous to the first result on the unit disk; that is, the weight decays at a polynomial rate on the boundary. Hence, it is natural to ask what happens if we take an exponentially decaying weight  $\lambda = \exp\left(\frac{1}{\rho}\right)$  in this setting. We present an answer in the following theorem.

**Theorem 1.** *Let  $\Omega$  be a smooth bounded complete Reinhardt domain in  $\mathbb{C}^2$  and let  $\rho$  be a smooth multi-radial defining function for  $\Omega$ . If  $\lambda = \exp\left(\frac{1}{\rho}\right)$  then the weighted Bergman projection  $\mathbf{B}_\Omega^\lambda$  is bounded on  $L^p(\Omega, \lambda)$  if and only if  $p = 2$ .*

Here, by a multi-radial function  $\rho(z_1, z_2)$  we mean a function that depends on  $|z_1|$  and  $|z_2|$ , i.e.  $\rho(z_1, z_2) = \rho(|z_1|, |z_2|)$ . The same result holds on smooth bounded complete Reinhardt domains in  $\mathbb{C}^n$  for  $n \geq 3$ . However, we will present the proof below only in two dimensions to avoid clumsy notation.

**1.2. Sobolev regularity on the unit ball.** Let  $\mathbb{B}^n$  denote the unit ball in  $\mathbb{C}^n$  and  $r(z_1, \dots, z_n) = |z_1|^2 + \dots + |z_n|^2 - 1$  be the standard defining function for  $\mathbb{B}^n$ . If we let  $\mu(z) = \exp\left(\frac{1}{r}\right)$  then by Theorem 1 the weighted Bergman projection  $\mathbf{B}_{\mathbb{B}^n}^\mu$  is not bounded on  $L^p(\mathbb{B}^n, \mu)$  for  $p \neq 2$ . Even though  $\mathbf{B}_{\mathbb{B}^n}^\mu$  exhibits this behavior on  $L^p$  spaces, its behavior is more predictable on  $L^2$ -Sobolev spaces. In particular, Sobolev regularity of  $\mathbf{B}_{\mathbb{B}^n}^\mu$  is not sensitive to decay rates. It is also noted in [CDM13, CDM14] that the weighted Bergman projection corresponding to the polynomially decaying weight  $(-r)^k$  is bounded on Sobolev spaces.

For  $k \in \mathbb{N}$ , we denote by  $W^k(\mathbb{B}^n, \mu)$  the weighted  $L^2$ -Sobolev space with the norm

$$\|f\|_{k, \mu}^2 = \sum_{|\beta + \gamma| \leq k} \int_{\mathbb{B}^n} \left| \frac{\partial^{\beta + \gamma}}{\partial \bar{z}^\beta \partial z^\gamma} f(z) \right|^2 \mu(z) dV(z).$$

For the exponentially decaying weight  $\mu$  on the unit ball, we prove the following boundedness result on weighted  $L^2$ -Sobolev spaces.

**Theorem 2.** *Let  $\mathbb{B}^n, r(z)$  and  $\mu$  be as above. For any  $k \in \mathbb{N}$ , the weighted Bergman projection  $\mathbf{B}_{\mathbb{B}^n}^\mu$  is bounded on  $W^k(\mathbb{B}^n, \mu)$ .*

It is a natural question to investigate the similar result for non-integer values of  $k$ . This boils down studying the interpolation properties of Sobolev spaces with respect to exponential weights. We postpone this investigation to a future project.

The unweighted version of this theorem on general smooth complete Reinhardt domains is in [Boa84] and [Str86]. We imitate the second proof in [Boa84]; however, instead of a Brunn-Minkowski type inequality we use an asymptotic estimate. It will be clear in the proof that generalizing Theorem 2 to general Reinhardt domains with general exponentially decaying weights is not immediate.

We write  $A \lesssim B$  to mean that there exists a uniform constant  $c > 0$  such that  $A \leq cB$ . We also use  $A \approx B$  to mean that there exists a uniform constant  $k > 0$  such that  $\frac{1}{k}A \leq B \leq kA$ .

## 2. PROOF OF THEOREM 1

**2.1. Preliminaries.** Since  $\Omega$  is a bounded Reinhardt domain, its projection onto  $z_1$  axis is a disk and without loss of generality we can assume that this projection is the unit disk. We denote the radial image of  $\Omega$  (that is the image in the  $|z_1|$  and  $|z_2|$  plane) by  $R \subset \mathbb{R}^2$ .

For a point  $z_1 \in \mathbb{D}$ , let  $S_{z_1}$  denote the slice of  $\Omega$  over  $z_1$  in the  $z_2$  direction. Each  $S_{z_1}$  is a disk. We define the following auxiliary weight on  $\mathbb{D}$ ,

$$(3) \quad \mu(z_1) = \text{area of the disk } S_{z_1} = \int_{S_{z_1}} \exp\left(\frac{1}{\rho}\right) dA(z_2)$$

where  $\rho$  is also restricted to  $S_{z_1}$  in the first coordinate. It is easy to notice that  $\mu$  is a multi-radial function of  $z_1$ .

The set of monomials  $\{z^\alpha\}$  (for all  $\alpha \in \mathbb{N}^2$ ) forms an orthogonal basis for the Bergman space  $L_a^2(\Omega, \lambda)$ . When we set coefficients

$$c_\alpha^2 = \frac{1}{\int_\Omega |z^\alpha|^2 \lambda(z) dV(z)},$$

we get the following representation of the weighted Bergman kernel (see [Kra01])

$$B_\Omega^\lambda(z, w) = \sum_\alpha c_\alpha^2 z^\alpha \overline{w}^\alpha.$$

**2.2. Reduction to  $B_\mathbb{D}^\mu$ .** We start with a relation between the weighted Bergman kernel  $B_\Omega^\lambda$  (defined on  $\Omega \times \Omega$ ) and the weighted Bergman kernel  $B_\mathbb{D}^\mu$  (defined on  $\mathbb{D} \times \mathbb{D}$ ).

**Lemma 4.** *For a fixed  $z \in \Omega$ ,*

$$(5) \quad \int_{S_{w_1}} B_\Omega^\lambda(z, w) \lambda(w) dA(w_2) = \mu(w_1) B_\mathbb{D}^\mu(z_1, w_1).$$

*Proof.* Since  $\Omega$  is Reinhardt, the Bergman kernel has the symmetry property that

$$B_\Omega^\lambda(z, w) = B_\Omega^\lambda(tz, t^{-1}w)$$

for  $t > 0$  such that  $tz$  and  $t^{-1}w$  are in  $\Omega$ . Hence when  $z$  is fixed, we can bump  $z$  out little bit and bump  $w$  in so that  $w$  is confined to a compact set. Now for  $w_1 \in \mathbb{D}$  we have

$$\int_{S_{w_1}} B_\Omega^\lambda(z, w) \lambda(w) dA(w_2) = \sum_\alpha c_\alpha^2 z^\alpha \overline{w_1}^{\alpha_1} \int_{S_{w_1}} \overline{w_2}^{\alpha_2} \exp\left(\frac{1}{\rho}\right) dA(w_2).$$

The change in order of summation and integration holds since for a fixed  $z$ , we can keep  $w$  in a compact set by the symmetry property above and the sum on the right hand side

converges uniformly. The function  $\exp\left(\frac{1}{\rho}\right)|_{w_1}$  is radially symmetric in  $w_2$ . Therefore, only  $\alpha_2 = 0$  contributes a non-zero integral above and we get

$$\begin{aligned} \int_{S_{w_1}} B_{\Omega}^{\lambda}(z, w) \lambda(w) dA(w_2) &= \sum_{\alpha_1=0}^{\infty} c_{(\alpha_1,0)}^2 z_1^{\alpha_1} \overline{w_1}^{\alpha_1} \int_{S_{w_1}} \exp\left(\frac{1}{\rho}\right) dA(w_2) \\ &= \mu(w_1) \sum_{\alpha_1=0}^{\infty} c_{(\alpha_1,0)}^2 z_1^{\alpha_1} \overline{w_1}^{\alpha_1}. \end{aligned}$$

We notice that

$$c_{(\alpha_1,0)}^2 = \frac{1}{\int_{\Omega} |z_1^{\alpha_1}|^2 \lambda(z) dV(z)} = \frac{1}{\int_{\mathbb{D}} |z_1^{\alpha_1}|^2 \mu(z_1) dA(z_1)},$$

and conclude that

$$\int_{S_{w_1}} B_{\Omega}^{\lambda}(z, w) \lambda(w) dA(w_2) = \mu(w_1) B_{\mathbb{D}}^{\mu}(z_1, w_1).$$

□

This observation enables us to prove the following relation between  $L^p$  mapping properties of weighted Bergman projections.

**Proposition 6.** *If the weighted Bergman projection  $\mathbf{B}_{\mathbb{D}}^{\mu}$  is unbounded on  $L^p(\mathbb{D}, \mu)$  for some  $p \geq 1$ , then so is  $\mathbf{B}_{\Omega}^{\lambda}$  on  $L^p(\Omega, \lambda)$ .*

*Proof.* Since the operator is unbounded there exists a sequence of functions  $\{f_n\} \subset L^p(\mathbb{D}, \mu) \cap L^2(\mathbb{D}, \mu)$  such that

$$(7) \quad \lim_{n \rightarrow \infty} \frac{\|\mathbf{B}_{\mathbb{D}}^{\mu} f_n\|_{\mathbb{D}, \mu}}{\|f_n\|_{\mathbb{D}, \mu}} = \infty.$$

The norms are  $L^p$  norms but we drop  $p$  to simplify the notation. We define a new sequence of functions  $\{F_n\} \subset L^p(\Omega, \lambda)$  by  $F_n(z_1, z_2) = f_n(z_1)$ . Then we get

$$(8) \quad \|F_n\|_{\Omega, \lambda} = \|f_n\|_{\mathbb{D}, \mu}.$$

On the other hand, by using (5) we notice that for any  $n \in \mathbb{N}$

$$(9) \quad \mathbf{B}_{\Omega}^{\lambda}(F_n)(z_1, z_2) = \mathbf{B}_{\mathbb{D}}^{\mu}(f_n)(z_1),$$

and by using (8)

$$(10) \quad \|\mathbf{B}_{\Omega}^{\lambda}(F_n)\|_{\Omega, \lambda} = \|\mathbf{B}_{\mathbb{D}}^{\mu}(f_n)\|_{\mathbb{D}, \mu}.$$

We finish the proof of the proposition by combining (7), (8) and (10).

□

**2.3. Analysis of  $\mathbf{B}_{\mathbb{D}}^{\mu}$ .** Next we study  $L^p$  mapping properties of the operator  $\mathbf{B}_{\mathbb{D}}^{\mu}$  and prove the following.

**Proposition 11.** *For  $1 < p < \infty$ ,  $\mathbf{B}_{\mathbb{D}}^{\mu}$  is bounded on  $L^p(\mathbb{D}, \mu)$  if and only if  $p = 2$ .*

The arguments are similar to the ones in [Zey13a]; however, the weights in [Zey13a] were explicit whereas the weight here is not. The weight on the unit disc here is the restriction of  $\lambda$ , which involves an unknown defining function in it.

*Proof.* It is enough to prove the statement for  $1 < p < 2$ . We get the remaining part by using duality of  $L^p$  spaces and self adjointness of  $\mathbf{B}_{\mathbb{D}}^{\mu}$ .

*Moment Function.* We define the following moment function (and its logarithm) for  $x > 0$ ,

$$\begin{aligned}\Phi(x) &= \int_{\mathbb{D}} |z|^x \mu(z) dA(z) \\ &= e^{\phi(x)}.\end{aligned}$$

After successive integration by parts  $n$ -times (due to exponential decay at the boundary we don't get any boundary terms), we rewrite it as

$$\begin{aligned}\Phi(x) &= \int_{\mathbb{D}} |z|^x \mu(z) dA(z) \\ &= \int_{\Omega} |z_1|^x \exp\left(\frac{1}{\rho}\right) dV(z_1, z_2) \\ &= 4\pi^2 \int_R r_1^{x+1} r_2 \exp\left(\frac{1}{\rho}\right) dr_1 dr_2 \\ &= \frac{4\pi^2}{(x+2) \cdots (x+n+1)} \int_R r_1^{x+n+1} (-1)^n \frac{\partial^n}{\partial r_1^n} \left( r_2 \exp\left(\frac{1}{\rho}\right) \right) dr_1 dr_2.\end{aligned}$$

We label the last integral and the part of the integrand as

$$\begin{aligned}\delta_n(r_1, r_2) &= (-1)^n \frac{\partial^n}{\partial r_1^n} \left( r_2 \exp\left(\frac{1}{\rho}\right) \right) \\ \Phi_n(x) &= \int_R r_1^{x+n+1} (-1)^n \frac{\partial^n}{\partial r_1^n} \left( r_2 \exp\left(\frac{1}{\rho}\right) \right) dr_1 dr_2 \\ &= \int_R r_1^{x+n+1} \delta_n(r_1, r_2) dr_1 dr_2.\end{aligned}$$

A straightforward computation expresses  $\delta_n$ 's as follows,

$$\begin{aligned}\delta_1(r_1, r_2) &= [\rho_{r_1}] \frac{r_2}{\rho^2} \exp\left(\frac{1}{\rho}\right) \\ \delta_2(r_1, r_2) &= [(\rho_{r_1})^2 + \rho(2(\rho_{r_1})^2 - \rho_{r_1 r_1} \rho)] \frac{r_2}{\rho^4} \exp\left(\frac{1}{\rho}\right) \\ &\vdots\end{aligned}$$

$$\delta_n(r_1, r_2) = [(\rho_{r_1})^n + \rho(\cdots)] \frac{r_2}{\rho^{2n}} \exp\left(\frac{1}{\rho}\right).$$

Without loss of generality,  $\rho_{r_1}$  can be assumed to be bounded away from zero near  $r_1 = 1$  since at  $r_1 = 1$  the tangent line is vertical. Moreover,  $\rho$  is negative inside, zero on the boundary, and positive outside; therefore the nonzero value of  $\rho_{r_1}$  has to be positive. In a small neighborhood of the boundary of  $R$ , where  $\rho$  is sufficiently small, only the first term  $(\rho_{r_1})^n$  in the square bracket matters since all the other terms contain a  $\rho$  term in front. That is, we obtain the following.

**Lemma 12.** *For any integer  $n$  there exists  $a_n > 0$  such that  $\delta_n(r_1, r_2) > c_n > 0$  for some  $c_n$  when  $a_n < r_1 < 1$  and for all  $r_2 > 0$ .*

Next we define

$$\tilde{\Phi}_n(x) = \int_{R \cap \{a_n < r_1 < 1\}} r_1^{x+n+1} \delta_n(r_1, r_2) dr_1 dr_2.$$

Then we compare  $\tilde{\Phi}_n(x)$  and  $\Phi_n(x)$ .

$$\begin{aligned} \left| \frac{\Phi_n(x)}{\tilde{\Phi}_n(x)} - 1 \right| &\leq \frac{\int_{R \cap \{r_1 < a_n\}} r_1^{x+n+1} |\delta_n(r_1, r_2)| dr_1 dr_2}{\int_{R \cap \{a_n < r_1 < 1\}} r_1^{x+n+1} \delta_n(r_1, r_2) dr_1 dr_2} \\ &\lesssim \frac{\max_R |\delta_n|}{\int_{R \cap \{a_n < r_1 < 1\}} \delta_n(r_1, r_2) dr_1 dr_2} \frac{a_n^{x+n+2}}{a_n^{x+n+1}(x+n+2)} \\ &\leq C(n) \frac{1}{x+n+2} \end{aligned}$$

Hence, we conclude the following lemma.

**Lemma 13.** *For any  $n \geq 1$ ,*

$$\lim_{x \rightarrow \infty} \frac{\Phi_n(x)}{\tilde{\Phi}_n(x)} = 1.$$

We define two more functions

$$\Theta_n(x) = \frac{1}{(x+2) \cdots (x+n+1)} \tilde{\Phi}_n(x)$$

and

$$\theta_n(x) = \log \Theta_n(x).$$

By Lemma 13 for any  $n \geq 1$ ,

$$(14) \quad \lim_{x \rightarrow \infty} \frac{\Theta_n(x)}{\Phi(x)} = 1$$

*Test Functions.* We fix once for all a positive integer  $k$  so large that  $p(k+1) < 2(k-1)$ . This is the step where we use the assumption  $p < 2$ . Then we consider the action of  $\mathbf{B}_{\mathbb{D}}^{\mu}$  on a sequence of monomials  $\{z^{km} \bar{z}^m\}_{m=0}^{\infty}$ . By using the radial symmetry, we obtain

$$\mathbf{B}_{\mathbb{D}}^{\mu}(z^{km} \bar{z}^m)(z) = \frac{\Phi(2km)}{\Phi(2(k-1)m)} z^{(k-1)m}.$$

In order to prove unboundedness, we consider the ratio

$$\begin{aligned} \frac{||\mathbf{B}_{\mathbb{D}}^{\mu}(z^{km}\bar{z}^m)||_{\mathbb{D},\mu}^p}{||z^{km}\bar{z}^m||_{\mathbb{D},\mu}^p} &= \left( \frac{\Phi(2km)}{\Phi(2(k-1)m)} \right)^p \frac{\Phi(p(k-1)m)}{\Phi(p(k+1)m)} \\ &= \exp(p\phi(2km) - p\phi(2(k-1)m) - (\phi(p(k+1)m) - \phi(p(k-1)m))). \end{aligned}$$

Our goal is to show that this expression gets as large as we want by choosing appropriate  $m$ .

In order to achieve this, we first look at another expression. We set

$$R_n(m) = p\theta_n(2km) - p\theta_n(2(k-1)m) - (\theta_n(p(k+1)m) - \theta_n(p(k-1)m)).$$

We note that the functions  $\Phi(x)$ ,  $\phi(x)$ ,  $\Theta_n(x)$  and  $\theta_n(x)$  are smooth functions of  $x$ . We use the mean value theorem twice to rewrite  $R_n(m)$  as follows

$$R_n(m) = 2pm\theta'_n(m_1) - 2pm\theta'_n(m_2)$$

for some  $m_1 \in (2(k-1)m, 2km)$  and  $m_2 \in (p(k-1)m, p(k+1)m)$ . Then

$$R_n(m) = 2pm(m_1 - m_2)\theta''_n(m_3)$$

for some  $m_3 \in (m_2, m_1)$ . Each of  $m_1$ ,  $m_2$  and  $m_3$  is comparable to  $m$  up to certain constants that depend on  $p$ . Therefore,

$$(15) \quad R_n(m) \geq cm_3^2\theta''_n(m_3) \quad \text{for some constant } c > 0.$$

*Second Derivative.* It remains to understand  $\theta''_n(m)$  as  $m$  goes to infinity. Recall the integration by parts argument at the beginning of the proof. When we take logarithm and two derivatives we get

$$\begin{aligned} \theta_n(x) &= \log \Theta_n(x) \\ \theta_n(x) &= -\log(x+2) - \dots - \log(x+n+1) + \log \tilde{\Phi}_n(x) \\ \theta''_n(x) &= \frac{1}{(x+2)^2} + \dots + \frac{1}{(x+n+1)^2} + \left( \log \tilde{\Phi}_n(x) \right)'' \\ \theta''_n(x) &\geq \frac{n}{(x+n+1)^2} + \left( \log \tilde{\Phi}_n(x) \right)'' \\ x^2\theta''_n(x) &\geq x^2 \frac{n}{(x+n+1)^2} + x^2 \left( \log \tilde{\Phi}_n(x) \right)'' . \end{aligned}$$

By Hölder's inequality, (since  $\delta_n$ 's are positive functions where we take integrals in  $\tilde{\Phi}_n$ )

$$\tilde{\Phi}_n(tu + (1-t)v) \leq \tilde{\Phi}_n(u)^t \tilde{\Phi}_n(v)^{1-t}.$$

In other words,  $\tilde{\Phi}_n(x)$  is log-convex<sup>†</sup> and  $\left( \log \tilde{\Phi}_n(x) \right)'' \geq 0$ . Hence we have the simpler inequality that for any  $n \geq 1$ ,

$$x^2\theta''_n(x) \geq x^2 \frac{n}{(x+n+1)^2}.$$

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<sup>†</sup>Since  $\delta_n$ 's are not necessarily positive on the whole  $R$  we can not claim  $\Phi_n(x)$  is log-convex.

We combine this with (15) to conclude that for any  $n \geq 1$ ,

$$R_n(m) \geq cm_3^2 \frac{n}{(m_3 + n + 1)^2}.$$

Recall  $m_3$  is comparable to  $m$ ; hence for any  $n$ , there exists  $M(n)$  such that for all  $m > M(n)$ ,

$$R_n(m) \geq c'n$$

for some constant  $c' > 0$ . When we take the exponential of both sides and recall the definition of  $R_n(m)$ , we get

$$\left( \frac{\Theta_n(2km)}{\Theta_n(2(k-1)m)} \right)^p \frac{\Theta_n(p(k-1)m)}{\Theta_n(p(k+1)m)} = \exp(R_n(m)) \geq \exp(c'n).$$

By using (14) and choosing  $m$  sufficiently large, we get

$$\left( \frac{\Phi(2km)}{\Phi(2(k-1)m)} \right)^p \frac{\Phi(p(k-1)m)}{\Phi(p(k+1)m)} \geq c'' \left( \frac{\Theta_n(2km)}{\Theta_n(2(k-1)m)} \right)^p \frac{\Theta_n(p(k-1)m)}{\Theta_n(p(k+1)m)} \geq c'' \exp(c' \frac{n}{2})$$

for some additional constant  $c'' > 0$ .

This means that there exist positive constants  $c'$  and  $c''$  such that for any  $n$ , there exists  $M'(n)$  so that for all  $m > M'(n)$

$$\left( \frac{\Phi(2km)}{\Phi(2(k-1)m)} \right)^p \frac{\Phi(p(k-1)m)}{\Phi(p(k+1)m)} \geq c'' \exp(c' \frac{n}{2})$$

Finally we let  $n$  go to infinity. This gives us the desired blowup and finishes the proof of Proposition 11.  $\square$

$\mathbf{B}_\Omega^\lambda$  is bounded on  $L^2(\Omega, \lambda)$  by definition. When we combine Propositions 6 and 11 we complete the proof of Theorem 1.

### 3. PROOF OF THEOREM 2

Recall that  $r(z_1, \dots, z_n) = |z_1|^2 + \dots + |z_n|^2 - 1$  is the standard defining function for  $\mathbb{B}^n$  and  $\mu(z) = \exp(\frac{1}{r})$ . For  $k \in \mathbb{N}$ , recall  $W^k(\mathbb{B}^n, \mu)$  denotes the weighted  $L^2$ -Sobolev space with the norm

$$\|f\|_{k,\mu}^2 = \sum_{|\beta+\gamma| \leq k} \int_{\mathbb{B}^n} \left| \frac{\partial^{\beta+\gamma}}{\partial \bar{z}^\beta \partial z^\gamma} f(z) \right|^2 \mu(z) dV(z).$$

For a multi-index  $\gamma$ , let  $d_\gamma$  denote the  $L^2$ -norm of the monomial  $z^\gamma$ , namely

$$d_\gamma^2 = \int_{\mathbb{B}^n} |z^\gamma|^2 \mu(z) dV(z).$$

For the proof of Theorem 2, we follow the argument in [Boa84, Section 4]. We start with the following lemmas. The first one is an asymptotic estimate that replaces a Brunn-Minkowski type inequality.



**Lemma 16.** *There exists a constant  $\kappa > 0$  such that*

$$(17) \quad \frac{1}{\kappa} \frac{e^{-2\sqrt{|\gamma|+1}} \gamma_1! \cdots \gamma_n!}{(|\gamma|+1)^{1/3} (|\gamma|+1)!} \leq d_\gamma^2 \leq \kappa \frac{e^{-2\sqrt{|\gamma|+1}} \gamma_1! \cdots \gamma_n!}{(|\gamma|+1)^{1/3} (|\gamma|+1)!}$$

for all multi-indices  $\gamma$ .

By using the estimate above, we obtain the following inequality between  $L^2$ -norms of monomials, which can be thought of as a reverse Cauchy-Schwarz inequality.

**Lemma 18.** *For a given multi-index  $\beta$  there exists a constant  $K_\beta$  such that*

$$(19) \quad d_\alpha d_{\alpha+2\beta} \leq K_\beta (d_{\alpha+\beta})^2$$

for all multi-indices  $\alpha$ .

The next two lemmas are generalizations of similar arguments in [Boa84] to the weighted setting.

**Lemma 20.** *For a given multi-index  $\beta$  there exists a bounded operator*

$$M_\beta : L_a^2(\mathbb{B}^n, \mu) \rightarrow L_a^2(\mathbb{B}^n, \mu)$$

such that

$$\left\langle h, \frac{\partial^\beta}{\partial z^\beta} g \right\rangle_\mu = \left\langle \frac{\partial^\beta}{\partial z^\beta} M_\beta h, g \right\rangle_\mu$$

for all holomorphic polynomials  $h$  and  $g \in L_a^2(\mathbb{B}^n, \mu)$ .

**Lemma 21.** *For a given multi-index  $\beta$  there exists a constant  $K_\beta$  such that*

$$\left| \left\langle \frac{\partial^\beta}{\partial z^\beta} h, f \right\rangle_\mu \right| \leq K_\beta \|h\|_{0,\mu} \|f\|_{|\beta|,\mu}$$

for all  $f \in W^{|\beta|}(\mathbb{B}^n, \mu)$  and all holomorphic polynomials  $h$ .

**3.1. Proof of Theorem 2.** For  $j \in \mathbb{N}$ , let  $S_j$  denote the truncation operator on  $L_a^2(\mathbb{B}^n, \mu)$ ; i.e. for  $f(z) = \sum_\alpha f_\alpha z^\alpha$

$$S_j f(z) = \sum_{|\alpha| \leq j} f_\alpha z^\alpha.$$

Note that  $S_j$  is a bounded operator with operator norm 1. Also as  $j \rightarrow \infty$ ,  $S_j f$  converges to  $f$  in norm.

For a given multi-index  $|\beta| \leq k$  and  $f \in W^k(\mathbb{B}^n, \mu)$  we want to show that

$$(22) \quad \left\| \frac{\partial^\beta}{\partial z^\beta} \mathbf{B}_\mu f \right\|_\mu^2 \lesssim \|f\|_{k,\mu}^2$$

where the constant is independent of  $f$ . This follows easily when we prove that

$$\left\| S_j \frac{\partial^\beta}{\partial z^\beta} \mathbf{B}_\mu f \right\|_\mu^2 \lesssim \|f\|_{k,\mu}^2$$

where the constant is independent of  $j$  and  $f$ .

Let  $h \in L_a^2(\mathbb{B}^n, \mu)$  and  $j \in \mathbb{N}$ , by using the lemmas above we obtain the following estimate

$$\begin{aligned}
\left| \left\langle h, S_j \frac{\partial^\beta}{\partial z^\beta} \mathbf{B}_\mu f \right\rangle_\mu \right| &= \left| \left\langle S_j h, \frac{\partial^\beta}{\partial z^\beta} \mathbf{B}_\mu f \right\rangle_\mu \right| \text{ by orthogonality of monomials} \\
&= \left| \left\langle \frac{\partial^\beta}{\partial z^\beta} M_\beta S_j h, f \right\rangle_\mu \right| \text{ by Lemma 20} \\
&\lesssim \|M_\beta S_j h\|_{0,\mu} \|f\|_{k,\mu} \text{ by Lemma 21} \\
&\lesssim \|h\|_{0,\mu} \|f\|_{k,\mu}
\end{aligned}$$

where the constant is independent of  $j$  and  $f$ . This estimate, with the duality of  $L^2$  spaces, proves (22) and we conclude the proof of Theorem 2 modulo proofs of the lemmas.

3.1.1. *Proof of Lemma 21.* This is essentially a repetition and combination of [Boa84, Lemma 6.1], [Str86, Lemma 2.1] and [Zey13b, Lemma 3]. We present the argument here for completeness.

If  $f$  is supported on a compact subset of  $\mathbb{B}^n$  then the estimate follows trivially. Hence we assume  $f$  is supported in a neighborhood of the boundary. We choose this neighborhood around the boundary such that there exist smoothly varying orthonormal holomorphic vector fields  $L_1, \dots, L_n$  such that  $L_1, \dots, L_{n-1}$  and  $L_n + \overline{L_n}$  are tangential to the boundary, see [Boa84, page 292]. In this case, by Cauchy-Riemann equations, for any  $1 \leq j \leq n$  there exist  $c_{ij} \in C^\infty(\overline{\mathbb{B}^n})$  such that

$$\frac{\partial}{\partial z_j} h = \left( c_{nj}(L_n + \overline{L_n}) + \sum_{i=1}^{n-1} c_{ij} L_i \right) h = \mathcal{L}_j h$$

for any holomorphic polynomial  $h$ . In other words, in this neighborhood of the boundary derivatives of a holomorphic polynomial can be written as a combination of tangential vector fields.

We know that for a tangential vector field  $T$ ,  $T(r) = 0$  where  $r$  is the standard defining function chosen at the beginning. Moreover, since  $\mu(z) = \exp(\frac{1}{r})$  we have  $T(\mu) = 0$  too. This means for  $f \in W^1(\mathbb{B}^n, \mu)$  and holomorphic polynomial  $h$ ,

$$\langle T(h), f \rangle_\mu = \langle T(h)\mu, f \rangle = \langle T(h\mu), f \rangle = \langle h\mu, \widetilde{T}(f) \rangle = \langle h, \widetilde{T}(f) \rangle_\mu$$

where  $\widetilde{T}$  is a first order differential operator with  $C^\infty(\overline{\mathbb{B}^n})$  coefficients. Note that no boundary term appears since  $T$  is tangential.

When we combine these two observations we get,

$$\begin{aligned}
\left\langle \frac{\partial}{\partial z_j} h, f \right\rangle_\mu &= \langle \mathcal{L}_j(h), f \rangle_\mu \text{ for some tangential vector field } \mathcal{L}_j \\
&= \langle h, \widetilde{\mathcal{L}}_j(f) \rangle_\mu
\end{aligned}$$

where  $\widetilde{\mathcal{L}}_j$  is a first order differential operator with  $C^\infty(\overline{\mathbb{B}^n})$  coefficients. Now iterating this several times, we get the following. For any multi-index  $\beta$ , holomorphic polynomial  $h$ , and

$$f \in W^{|\beta|}(\mathbb{B}^n, \mu)$$

$$\left\langle \frac{\partial^\beta}{\partial z^\beta} h, f \right\rangle_\mu = \left\langle h, \widetilde{\mathcal{L}}_\beta(f) \right\rangle_\mu$$

for some differential operator  $\widetilde{\mathcal{L}}_\beta$  of order  $|\beta|$  with  $C^\infty(\overline{\mathbb{B}^n})$  coefficients. Now, Lemma 21 follows from the Cauchy-Schwarz inequality.

3.1.2. *Proof of Lemma 20.* This lemma follows from Lemma 18 once we define  $M_\beta$  as follows. For a monomial  $z^\alpha$ ,

$$M_\beta(z^\alpha) = \frac{(\alpha + \beta)!(\alpha + \beta)!}{\alpha!(\alpha + 2\beta)!} \frac{d_\alpha^2}{d_{\alpha+\beta}^2} z^{\alpha+2\beta}.$$

Then

$$\frac{\|M_\beta(z^\alpha)\|_\mu}{\|z^\alpha\|_\mu} = \frac{(\alpha + \beta)!(\alpha + \beta)!}{\alpha!(\alpha + 2\beta)!} \frac{d_\alpha d_{\alpha+2\beta}}{d_{\alpha+\beta}^2}.$$

The first fraction is uniformly bounded since

$$\frac{(\alpha + \beta)!(\alpha + \beta)!}{\alpha!(\alpha + 2\beta)!} = \frac{\binom{\alpha+\beta}{\beta}}{\binom{\alpha+2\beta}{\beta}} \leq 1.$$

The second fraction is uniformly bounded by Lemma 18. Since monomials form an orthogonal basis for  $L_a^2(\mathbb{B}^n, \mu)$  we conclude the proof.

We note that the proofs of both Lemmas 21 and 20 work not only for the particular  $(\mathbb{B}^n, \mu)$  pair but also for any smooth complete Reinhardt domain and any exponentially decaying weight. However, the estimate (19) on the coefficients requires more work and it is the heart of the matter. It follows from the estimate (17) and we present the proof of (17) on the unit ball with the specific weight  $\mu$ . Readers will note that similar arguments work on some more general domains (e.g. complex ellipsoids). However, obtaining this sort of decay rate on the  $L^2$ -norms of monomials is not immediate on general smooth Reinhardt domains with exponentially decaying weights. We leave the general discussion of all complete Reinhardt domains with exponential weights to a future work.

3.1.3. *Proof of Lemma 16.* We start with the following estimate, see [Dos04, Lemma 2.2] and [Dos07, Lemma 1]. As  $x \rightarrow \infty$ ,

$$(23) \quad \int_0^1 r^x \exp\left(\frac{-1}{1-r}\right) dr \approx x^{-1/3} e^{-2\sqrt{x}}.$$

We obtain (17) by taking integrals in radial coordinates and using the estimate (23). We go over the details for the case  $n = 2$ , the general case follows similarly. We denote the radial image of  $\mathbb{B}^2$ , the quarter circle in the first quadrant, by  $\mathcal{R}$ .

$$\begin{aligned} d_\gamma^2 &= \int_{\mathbb{B}^2} |z^\gamma|^2 \mu(z) dV(z) \\ &= 4\pi^2 \int_{\mathcal{R}} r_1^{2\gamma_1+1} r_2^{2\gamma_2+1} \exp\left(\frac{-1}{1-(r_1^2+r_2^2)}\right) dr_2 dr_1 \end{aligned}$$

where  $z_1 = r_1 e^{i\theta_1}$  and  $z_2 = r_2 e^{i\theta_2}$ . Next, we take the integral over  $\mathcal{R}$  in radial coordinates. We set  $r_1 = R \cos \theta$  and  $r_2 = R \sin \theta$ .

$$\begin{aligned} d_\gamma^2 &= 4\pi^2 \int_0^1 \int_0^{\pi/2} (R \cos \theta)^{2\gamma_1+1} (R \sin \theta)^{2\gamma_2+1} \exp\left(\frac{-1}{1-R^2}\right) R dR d\theta \\ &= 4\pi^2 \int_0^1 R^{2\gamma_1+2\gamma_2+3} \exp\left(\frac{-1}{1-R^2}\right) dR \int_0^{\pi/2} (\cos \theta)^{2\gamma_1+1} (\sin \theta)^{2\gamma_2+1} d\theta \\ &= 2\pi^2 \int_0^1 R^{|\gamma|+1} \exp\left(\frac{-1}{1-R^2}\right) dR \int_0^{\pi/2} (\cos \theta)^{2\gamma_1+1} (\sin \theta)^{2\gamma_2+1} d\theta \end{aligned}$$

A simple calculation of trigonometric integrals reveals that

$$\int_0^{\pi/2} (\cos \theta)^{2\gamma_1+1} (\sin \theta)^{2\gamma_2+1} d\theta = \frac{1}{2} \frac{\gamma_1! \gamma_2!}{(|\gamma|+1)!}.$$

Therefore, for big enough  $|\gamma|$ , we invoke (23) to get

$$d_\gamma^2 \approx \frac{e^{-2\sqrt{|\gamma|+1}}}{(|\gamma|+1)^{1/3}} \frac{\gamma_1! \gamma_2!}{(|\gamma|+1)!}$$

where the constant is independent of the multi-index  $\gamma$ . This estimate concludes the proof of Lemma 16.

Once we have the asymptotic behavior of  $L^2$ -norms of monomials the next proof follows quickly.

3.1.4. *Proof of Lemma 18.* We remark again that the inequality (19) (constant independent of  $\alpha$ ) is not immediate. We plug (17) back in (19),

$$\begin{aligned} \left( \frac{d_\alpha d_{\alpha+2\beta}}{d_{\alpha+\beta}^2} \right)^2 &\approx \frac{e^{-2\sqrt{|\alpha|+1}}}{(|\alpha|+1)^{1/3}} \frac{\alpha_1! \alpha_2!}{(|\alpha|+1)!} \frac{e^{-2\sqrt{|\alpha+2\beta|+1}}}{(|\alpha+2\beta|+1)^{1/3}} \frac{(\alpha_1+2\beta_1)! (\alpha_2+2\beta_2)!}{(|\alpha+2\beta|+1)!} \\ &\times \left( \frac{(|\alpha+\beta|+1)^{1/3}}{e^{-2\sqrt{|\alpha+\beta|+1}}} \frac{(|\alpha+\beta|+1)!}{(\alpha_1+\beta_1)! (\alpha_2+\beta_2)!} \right)^2 \\ &\approx \exp \left[ -2 \left( \sqrt{|\alpha|+1} + \sqrt{|\alpha+2\beta|+1} - 2\sqrt{|\alpha+\beta|+1} \right) \right] \\ &\times \frac{(|\alpha+\beta|+1)^{2/3}}{(|\alpha|+1)^{1/3} (|\alpha+2\beta|+1)^{1/3}} \\ &\times \frac{\alpha_1! \alpha_2! (\alpha_1+2\beta_1)! (\alpha_2+2\beta_2)!}{(\alpha_1+\beta_1)! (\alpha_1+\beta_1)! (\alpha_2+\beta_2)! (\alpha_2+\beta_2)!} \\ &\times \frac{(|\alpha+\beta|+1)! (|\alpha+\beta|+1)!}{(|\alpha|+1)! (|\alpha+2\beta|+1)!} \end{aligned}$$

The three fractions after the exponential term are uniformly bounded for all  $\alpha$ . For the first fraction this is immediate and for the next two we can use the Stirling's formula. If we make sure that the expression inside the exponential function

$$(24) \quad -2 \left( \sqrt{|\alpha|+1} + \sqrt{|\alpha+2\beta|+1} - 2\sqrt{|\alpha+\beta|+1} \right)$$

is uniformly bounded then everything will be uniformly bounded too. Indeed,

$$\begin{aligned} -2 \left( \sqrt{|\alpha|+1} + \sqrt{|\alpha+2\beta|+1} - 2\sqrt{|\alpha+\beta|+1} \right) &\leq 2 \left( \sqrt{|\alpha+\beta|+1} - \sqrt{|\alpha|+1} \right) \\ &= \frac{2|\beta|}{\sqrt{|\alpha+\beta|+1} + \sqrt{|\alpha|+1}} \\ &\leq 2|\beta|. \end{aligned}$$

Therefore (24) is uniformly bounded. Hence we obtain (19) and conclude the proof of Lemma 18.

By this we conclude the proof of Theorem 2. The proof of the key Lemma 18 utilized the asymptotic information (17), instead of a Brunn-Minkowski type inequality. For the analog of Theorem 2 on convex Reinhardt domains, such an inequality is needed.

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